NULL CONTROLLABILITY OF ONE-DIMENSIONAL PARABOLIC EQUATIONS BY THE FLATNESS APPROACH

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ABSTRACT. We consider linear one-dimensional parabolic equations with space dependent coefficients that are only measurable and that may be degenerate or singular. Considering generalized Robin-Neumann boundary conditions at both extremities, we prove the null controllability with one boundary control by following the flatness approach, which provides explicitly the control and the associated trajectory as series. Both the control and the trajectory have a Gevrey regularity in time related to the L^p class of the coefficient in front of u_t. The approach applies in particular to the (possibly degenerate or singular) heat equation $(a(x)u_x)_x - u_t = 0$ with $a(x) > 0$ for a.e. $x \in (0,1)$ and $a + 1/a \in L^1(0,1)$, or to the heat equation with inverse square potential $u_{xx} + (\mu/|x|^2)u - u_t = 0$ with $\mu \geq 1/4$.

1. Introduction

The null controllability of parabolic equations has been extensively investigated since several decades. After the pioneering work in [15, 23, 30], mainly concerned with the one-dimensional case, there has been significant progress in the general N-dimensional case [18, 22, 29] by using Carleman estimates. The more recent developments of the theory were concerned with discontinuous coefficients [2, 4, 16, 28], degenerate coefficients [1, 3, 7, 8, 9, 10, 11, 17], or singular coefficients [12, 14, 38].

In [2], the authors derived the null controllability of a linear one-dimensional parabolic equation with (essentially bounded) measurable coefficients. The method of proof combined the Lebeau-Robbiano approach [29] with some complex analytic arguments.

Here, we are concerned with the null controllability of the system

$$\begin{aligned}
(a(x)u_x)_x + b(x)u_x + c(x)u - \rho(x)u_t &= 0, \quad x \in (0,1),\ t \in (0,T), && (1.1)\\
\alpha_0 u(0,t) + \beta_0 (au_x)(0,t) &= 0, \quad t \in (0,T), && (1.2)\\
\alpha_1 u(1,t) + \beta_1 (au_x)(1,t) &= h(t), \quad t \in (0,T), && (1.3)\\
u(x,0) &= u_0(x), \quad x \in (0,1), && (1.4)
\end{aligned}$$

where $(\alpha_0, \beta_0), (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ are given, $u_0 \in L^2(0,1)$ is the initial state and $h \in L^2(0,T)$ is the control input.

Key words and phrases. degenerate parabolic equation; singular coefficient; null controllability, Gevrey functions; flatness.

The given functions a, b, c, ρ will be assumed to fulfill the following conditions

$$a(x) > 0 \text{ and } \rho(x) > 0 \text{ for a.e. } x \in (0,1), \tag{1.5}$$

$$(\frac{1}{a}, \frac{b}{a}, c, \rho) \in [L^1(0,1)]^4, \tag{1.6}$$

$$\exists K \geq 0, \quad \frac{c(x)}{\rho(x)} \leq K \quad \text{for a.e. } x \in (0,1), \tag{1.7}$$

$$\exists p \in (1, \infty], \quad a^{1-\frac{1}{p}} \rho \in L^p(0,1). \tag{1.8}$$

The assumptions (1.5)-(1.8) are clearly less restrictive than the assumptions from [2]:

$$a, b, c, \rho \in L^\infty(0,1) \text{ and } a(x) > \varepsilon, \ \rho(x) > \varepsilon > 0 \text{ for a.e. } x \in (0,1) \tag{1.9}$$

for some $\varepsilon > 0$.

Let us introduce some notations. Let B be a Banach space with norm $\| \cdot \|_B$. For any $t_1 < t_2$ and $s \geq 0$, we denote by $G^s([t_1, t_2], B)$ the class of (Gevrey) functions $u \in C^\infty([t_1, t_2], B)$ for which there exist positive constants M, R such that

$$\|u^{(p)}(t)\|_B \leq M \frac{p!^s}{R^p} \quad \forall t \in [t_1, t_2], \ \forall p \geq 0. \tag{1.10}$$

When $(B, \| \cdot \|_B) = (\mathbb{R}, | \cdot |)$, $G^s([t_1, t_2], B)$ is merely denoted $G^s([t_1, t_2])$. Let

$$L^1_\rho := \{u : (0,1) \to \mathbb{R}; \ \ ||u||_{L^1_\rho} := \int_0^1 |u(x)| \rho(x) dx < \infty \}.$$

Note that $L^2(0,1) \subset L^1_\rho$ if $\rho \in L^2(0,1)$. The main result in this paper is the following

Theorem 1.1. *Let the functions $a, b, c, \rho : (0,1) \to \mathbb{R}$ satisfy (1.5)-(1.8) for some numbers $K \geq 0$, $p \in (1, \infty]$. Let $(\alpha_0, \beta_0), (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $T > 0$. Pick any $u_0 \in L^1_\rho$ and any $s \in (1, 2 - 1/p)$. Then there exists a function $h \in G^s([0,T])$, that may be given explicitly as a series, such that the solution u of (1.1)-(1.4) satisfies $u(., T) = 0$. Moreover $u \in G^s([\varepsilon, T], W^{1,1}(0,1))$ and $au_x \in G^s([\varepsilon, T], C^0([0,1]))$ for all $\varepsilon \in (0,T)$.*

Clearly, Theorem 1.1 can be applied to parabolic equations with discontinuous coefficients that may be degenerate or singular at a point (or more generally at a sequence of points). The proof of it is not based on the classical duality approach, in the sense that it does not rely on the proof of some observability inequality for the adjoint equation. It follows the flatness approach developed in [25, 26, 27, 31, 32, 33, 35]. This direct approach gives explicitly both the control and the trajectory as series, which leads to efficient numerical schemes by taking partial sums in the series [33, 34]. Let us describe its main steps. In the first step, following [2], we show that after a series of changes of dependent/independent variables, system (1.1)-(1.4) can be put into the canonical form

$$\begin{aligned}
u_{xx} - \rho(x) u_t &= 0, \quad x \in (0,1), \ t \in (0,T), && (1.11)\\
\alpha_0 u(0,t) + \beta_0 u_x(0,t) &= 0, \quad t \in (0,T), && (1.12)\\
\alpha_1 u(1,t) + \beta_1 u_x(1,t) &= h(t), \quad t \in (0,T), && (1.13)\\
u(x,0) &= u_0(x), \quad x \in (0,1), && (1.14)
\end{aligned}$$

where $\rho(x) > 0$ a.e. in $(0,1)$ and $\rho \in L^p(0,1)$ with $p \in (1,\infty]$. In the second step, following [31, 33], we seek u in the form

$$u(x,t) = \sum_{n\ge 0} e^{-\lambda_n t} e_n(x), \quad x \in (0,1),\ t \in [0,\tau], \tag{1.15}$$

$$u(x,t) = \sum_{i\ge 0} y^{(i)}(t) g_i(x), \quad x \in (0,1),\ t \in [\tau,T], \tag{1.16}$$

where $\tau \in (0,T)$ is any intermediate time; $e_n : (0,1) \to \mathbb{R}$ (resp. $\lambda_n \in \mathbb{R}$) denotes the n^{th} *eigenfunction* (resp. *eigenvalue*) associated with (1.11)-(1.13) and satisfying [25, 26]

$$-e_n'' = \lambda_n \rho e_n, \quad x \in (0,1) \tag{1.17}$$

$$\alpha_0 e_n(0) + \beta_0 e_n'(0) = 0, \tag{1.18}$$

$$\alpha_1 e_n(1) + \beta_1 e_n'(1) = 0, \tag{1.19}$$

while $g_i : (0,1) \to \mathbb{R}$ is defined inductively as the solution to the Cauchy problem

$$g_0'' = 0, \quad x \in (0,1) \tag{1.20}$$

$$\alpha_0 g_0(0) + \beta_0 g_0'(0) = 0, \tag{1.21}$$

$$\beta_0 g_0(0) - \alpha_0 g_0'(0) = 1 \tag{1.22}$$

for $i = 0$, and to the Cauchy problem

$$g_i'' = \rho g_{i-1}, \quad x \in (0,1) \tag{1.23}$$

$$g_i(0) = 0, \tag{1.24}$$

$$g_i'(0) = 0 \tag{1.25}$$

for $i \ge 1$. Expanding u on generating functions as in (1.16) rather than on powers of x as in [27, 31] was introduced in [26] and studied in [25].

The fact that the generating function g_i is defined as the solution of a *Cauchy problem*, rather than the solution of a *boundary-value problem*, is crucial in the analysis developed here. First, it allows to prove that *every* initial state in the space L^1_ρ (and not only states in some restricted class of Gevrey functions) can be driven to 0 in time T. Secondly, from (1.23)-(1.25), we see by an easy induction on i that for $\rho \in L^\infty(0,1)$, the function g_i is uniformly bounded by $C/(2i)!$, and hence the series in (1.16) is indeed convergent when $y \in G^s([\tau,T])$ with $1 < s < 2$.

The corresponding control function h is given explicitly as

$$h(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tau, \\ \sum_{i\ge 0} y^{(i)}(t)(\alpha_1 g_i(1) + \beta_1 g_i'(1)) & \text{if } \tau < t \le T. \end{cases}$$

It is easy to see that the function $u(x,t)$ defined in (1.16) satisfies (formally) (1.11), and also the condition $u(x,T) = 0$ if $y^{(i)}(T) = 0$ for all $i \in \mathbb{N}$, so that the null controllability can be established for *some* initial states. The main issue is then to extend it to *every* initial state $u_0 \in L^1_\rho$. Following [31, 32, 33], we first use the strong smoothing effect of the heat equation to smooth out the state function in the time interval $(0,\tau)$. Next, to ensure that the two expressions of u given in (1.15)-(1.16) coincide at $t = \tau$, we have to relate the eigenfunctions e_n to the generating functions g_i.

It will be shown that any eigenfunction e_n can be expanded in terms of the generating functions g_i as

$$e_n(x) = \zeta_n \sum_{i\geq 0} (-\lambda_n)^i g_i(x) \tag{1.26}$$

with $\zeta_n \in \mathbb{R}$. Note that, for $\rho \equiv 1$ and $(\alpha_0, \beta_0, \alpha_1, \beta_1) = (0,1,0,1)$, $\lambda_n = (n\pi)^2$ for all $n \geq 0$, $e_0(x) = 1$ and $e_n(x) = \sqrt{2}\cos(n\pi x)$ for $n \geq 0$ while $g_i(x) = x^{2i}/(2i)!$, so that (1.26) for $n \geq 1$ is nothing but the classical Taylor expansion of $\cos(n\pi x)$ around $x = 0$:

$$\cos(n\pi x) = \sum_{i\geq 0} (-1)^i \frac{(n\pi x)^{2i}}{(2i)!}\cdot \tag{1.27}$$

Thus (1.26) can be seen as a natural extension of (1.27), in which the generating functions g_i, *a priori* not smoother than $W^{2,p}(0,1)$, replace the functions $x^{2i}/(2i)!$.

The condition (1.8) is used to prove the estimate

$$|g_i(x)| \leq \frac{C}{R^{2i}(i!)^{2-\frac{1}{p}}}$$

needed to ensure the convergence of the series in (1.16) when $y \in G^s([\tau, T])$ with $1 < s < 2-1/p$.

Theorem 1.1 applies in particular to any system

$$\begin{aligned}
(a(x)u_x)_x - u_t &= 0, \quad x \in (0,1),\ t \in (0,T), &(1.28)\\
\alpha_0 u(0,t) + \beta_0 (au_x)(0,t) &= 0, \quad t \in (0,T), &(1.29)\\
\alpha_1 u(1,t) + \beta_1 (au_x)(1,t) &= h(t), \quad t \in (0,T), &(1.30)\\
u(x,0) &= u_0(x), \quad x \in (0,1), &(1.31)
\end{aligned}$$

where $a(x) > 0$ for a.e. $x \in (0,1)$ and $a + 1/a \in L^1(0,1)$. (Pick $p = 2$ in (1.8).) This includes the case where a is measurable, positive and essentially bounded together with its inverse (but not necessarily piecewise continuous), and the case where $a(x) = x^r$ with $-1 < r < 1$. (Actually any $r \leq -1$ is also admissible, by picking $p > 1$ sufficiently close to 1 in (1.8).) Note that our result applies as well to $a(x) = (1-x)^r$ with $0 < r < 1$, yielding a positive null controllability result when the control is applied at the point $(x = 1)$ where the diffusion coefficient degenerates (see [10, Section 2.7]). Note also that the coefficient $a(x)$ is allowed to be degenerate/singular at a *sequence* of points: consider e.g. $a(x) := |\sin(x^{-1})|^r$ with $-1 < r < 1$. Then $a + 1/a \in L^1(0,1)$.

The null controllability of (1.28)-(1.31) for $a(x) = x^r$ with $0 < r < 2$ was established (in appropriate spaces) in [10]. The situation when $1/a \notin L^1(0,1)$ (e.g. $a(x) = x^r$ with $1 \leq r < 2$) is beyond these notes, and it will be considered elsewhere.

A null controllability result with an internal control can be deduced from (1.1). Its proof is given in appendix, for the sake of completeness.

Corollary 1.2. *Assume given an open set $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < 1$, and let us consider the following control system*

$$
\begin{aligned}
(a(x)u_x)_x + b(x)u_x + c(x)u - \rho(x)u_t &= \chi_\omega f(x,t), \quad x \in (0,1),\ t \in (0,T), &(1.32)\\
\alpha_0 u(0,t) + \beta_0 (au_x)(0,t) &= 0, \quad t \in (0,T), &(1.33)\\
\alpha_1 u(1,t) + \beta_1 (au_x)(1,t) &= 0, \quad t \in (0,T), &(1.34)\\
u(x,0) &= u_0(x), \quad x \in (0,1), &(1.35)
\end{aligned}
$$

where $u_0 \in L^1_\rho$ is any given initial data, and a, b, c, ρ, p, K, s, (α_0, β_0) and (α_1, β_1) are as in Theorem 1.1. Then one can find a control input $f \in L^2(0, T, L^2_a(\omega))$ such that the solution u of (1.32)-(1.35) satisfies $u(x,T) = 0$ for all $x \in (0,1)$.

Another important family of heat equations with variable coefficients is those with inverse square potential localized at the boundary, namely

$$
\begin{aligned}
u_{xx} + \frac{\mu}{x^2}u - u_t &= 0, \quad x \in (0,1),\ t \in (0,T), &(1.36)\\
u(0,t) &= 0, \quad t \in (0,T), &(1.37)\\
\alpha_1 u(1,t) + \beta_1 u_x(1,t) &= h(t), \quad t \in (0,T), &(1.38)\\
u(x,0) &= u_0(x), \quad x \in (0,1), &(1.39)
\end{aligned}
$$

where $\mu \in \mathbb{R}$ is a given number. Note that Theorem 1.1 cannot be applied to (1.36)-(1.39), for $c(x) = \mu x^{-2}$ is not integrable on $(0,1)$. It was proved in [12] that (1.36)-(1.39) is null controllable in $L^2(0,1)$ when $\mu \le 1/4$ by combining Carleman inequalities to Hardy inequalities. We shall show in this paper that this result can be retrieved by the flatness approach as well.

Theorem 1.3. *Let $\mu \in (0, 1/4]$, $(\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $T > 0$, and $\tau \in (0,T)$. Pick any $u_0 \in L^2(0,1)$ and any $s \in (1,2)$. Then there exists a function $h \in G^s([0,T])$ with $h(t) = 0$ for $0 \le t \le \tau$ and such that the solution u of (1.36)-(1.39) satisfies $u(T,.) = 0$. Moreover, $u \in G^s([\varepsilon, T], W^{1,1}(0,1))$ for all $\varepsilon \in (0,T)$. Finally, if $0 \le \mu < 1/4$ and $r > (1 + \sqrt{1-4\mu})/2$, then $x^r u_x \in G^s([\varepsilon, T], C^0([0,1]))$ for all $\varepsilon \in (0,T)$.*

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. We first show that a convenient change of variables transforms (1.1)-(1.4) into (1.11)-(1.14) (Proposition 2.1). Next, we show that the flatness approach can be applied to (1.11)-(1.14) to yield a null controllability result (Theorem 2.9). Performing the inverse change of variables, we complete the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.3, which is obtained as a consequence of Theorem 2.9 after a convenient change of variables, and some examples.

2. Proof of Theorem 1.1

2.1. **Reduction to the canonical form (1.11)-(1.14).** Let a, b, c, ρ, and p be as in (1.5)-(1.8). Set

$$
\begin{aligned}
B(x) &:= \int_0^x \frac{b(s)}{a(s)} ds, &(2.1)\\
\tilde{a}(x) &:= a(x)e^{B(x)} &(2.2)\\
\tilde{c}(x) &:= (K\rho(x) - c(x))e^{B(x)}. &(2.3)
\end{aligned}
$$

Then $B \in W^{1,1}(0,1)$, $\tilde{c} \in L^1(0,1)$, and

$$\tilde{a}(x) > 0 \text{ and } \tilde{c}(x) \geq 0 \text{ for a.e. } x \in (0,1).$$

We introduce the solution v to the elliptic boundary value problem

$$-(\tilde{a}v_x)_x + \tilde{c}v = 0, \quad x \in (0,1), \tag{2.4}$$

$$v(0) = v(1) = 1, \tag{2.5}$$

and set

$$u_1(x,t) := e^{-Kt}u(x,t), \tag{2.6}$$

$$u_2(x,t) := \frac{u_1(x,t)}{v(x)}. \tag{2.7}$$

Finally, let

$$L := \int_0^1 (a(s)v^2(s)e^{B(s)})^{-1}ds, \quad y(x) := \frac{1}{L}\int_0^x (a(s)v^2(s)e^{B(s)})^{-1}ds \tag{2.8}$$

and

$$\hat{u}(y,t) := u_2(x,t), \qquad \hat{\rho}(y) := L^2 a(x)v^4(x)e^{2B(x)}\rho(x) \tag{2.9}$$

for $0 < t < T$, $y = y(x)$ with $x \in [0,1]$. Then the following result holds.

Proposition 2.1. (i) *$v \in W^{1,1}(0,1)$ and $0 < v(x) \leq 1$ $\forall x \in [0,1]$;*
(ii) *$y : [0,1] \to [0,1]$ is an increasing bijection with $y, y^{-1} \in W^{1,1}(0,1)$;*
(iii) *$\hat{\rho}(y) > 0$ for a.e. $y \in (0,1)$, and $\hat{\rho} \in L^p(0,1)$;*
(iv) *$\hat{u}$ solves the system*

$$\hat{u}_{yy} - \hat{\rho}\hat{u}_t = 0, \quad y \in (0,1),\ t \in (0,T), \tag{2.10}$$

$$\hat{\alpha}_0\hat{u}(0,t) + \hat{\beta}_0\hat{u}_y(0,t) = 0, \quad t \in (0,T), \tag{2.11}$$

$$\hat{\alpha}_1\hat{u}(1,t) + \hat{\beta}_1\hat{u}_y(1,t) = \hat{h}(t) := e^{-Kt}h(t), \quad t \in (0,T), \tag{2.12}$$

$$\hat{u}(y(x),0) = \frac{u_0(x)}{v(x)}, \quad x \in (0,1), \tag{2.13}$$

for some $(\hat{\alpha}_0, \hat{\beta}_0), (\hat{\alpha}_1, \hat{\beta}_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Proof. (i) Let $l = \int_0^1 ds/\tilde{a}(s)$ and $z(x) = l^{-1}\int_0^x ds/\tilde{a}(s)$. Then $z : [0,1] \to [0,1]$ is a strictly increasing continuous map (for $z(x_2) - z(x_1) = l^{-1}\int_{x_1}^{x_2} ds/\tilde{a}(s) > 0$ for $x_1 < x_2$). It is a bijection which is absolutely continuous (i.e. $z \in W^{1,1}(0,1)$), for $1/\tilde{a} \in L^1(0,1)$. Moreover, $z'(x) = 1/(l\tilde{a}(x))$ for a.e. $x \in (0,1)$. It follows from (1.5) and (1.8) that $a(x) < \infty$ and $\tilde{a}(x) < \infty$ for a.e. $x \in (0,1)$, so that $z'(x) > 0$ for a.e. $x \in (0,1)$. Then we infer from a theorem due to M. A. Zareckii (see [6, Ex. 5.8.54 p. 389] or [37]) that z^{-1} is absolutely continuous as well (i.e. $z^{-1} \in W^{1,1}(0,1)$). (Note that for $z : [0,1] \to [0,1]$ a strictly increasing bijection in $W^{1,1}(0,1)$, its inverse z^{-1} may not belong to $W^{1,1}(0,1)$, see [19, Ex. 4.6 p. 287] or [37].) In particular, z satisfies the condition N (Lusin's condition)

$$A \subset [0,1],\ |A| = 0 \Rightarrow |z(A)| = 0 \tag{2.14}$$

($|A|$ standing for the Lebesgue measure of A), and the same holds true for z^{-1}.

Introduce the function $w : [0, 1] \to \mathbb{R}$ defined by

$$w(z) := v(x(z)) \qquad \forall z \in [0, 1].$$

Then $dw/dz = l\tilde{a}(x)dv/dx$ so that, letting $' = d/dz$ and $\gamma(z) := (l^2\tilde{a}\tilde{c})(x(z))$, (2.4)-(2.5) becomes

$$-w'' + \gamma w = 0, \quad z \in (0, 1) \tag{2.15}$$

$$w(0) = w(1) = 1. \tag{2.16}$$

Note that $\gamma(z) \geq 0$ for a.e. $z \in (0, 1)$ and that $\gamma \in L^1(0, 1)$, for

$$\int_0^1 \gamma(z)dz = l \int_0^1 \tilde{c}(x(z))\frac{dx}{dz}dz = l \int_0^1 \tilde{c}(x)dx < \infty.$$

In the last equality, we used the change of variable formula (which is licit, because $z^{-1} \in W^{1,1}(0, 1)$ and it satisfies Lusin's condition, see [20]). Letting $w = u + 1$, we define u as the unique solution in $H_0^1(0, 1)$ of the variational problem

$$\int_0^1 [u'\varphi' + \gamma u\varphi]dx = -\int_0^1 \gamma\varphi dx \quad \forall \varphi \in H_0^1(0, 1).$$

Then $w \in W^{2,1}(0, 1) \subset C^1([0, 1])$. Let us check that

$$0 < w(x) \leq 1 \quad \forall x \in [0, 1]. \tag{2.17}$$

If $\max_{x\in[0,1]} w(x) > 1$, we can pick $x_0 \in (0, 1)$ such that

$$w(x_0) = \max_{x\in[0,1]} w(x) > 1. \tag{2.18}$$

Then $w'(x_0) = 0$. Let $\delta > 0$ denote the greatest positive number such that $x_0 + \delta \leq 1$ and

$$w(x) > 1 \qquad \forall x \in (x_0, x_0 + \delta).$$

It follows that for $x \in [x_0, x_0 + \delta]$

$$w'(x) = \int_{x_0}^x w''(s)ds = \int_{x_0}^x \gamma(s)w(s)ds \geq 0$$

and hence

$$w(x) - w(x_0) = \int_{x_0}^x w'(s)ds \geq 0.$$

In particular, $w(x_0 + \delta) \geq w(x_0) > 1$, a fact which contradicts the definition of δ. Thus $\max_{x\in[0,1]} w(x) \leq 1$. A similar argument shows that $\min_{x\in[0,1]} w(x) \geq 0$. If $\min_{x\in[0,1]} w(x) = 0$, we pick $x_0 \in (0, 1)$ such that

$$w(x_0) = \min_{x\in[0,1]} w(x) = 0.$$

Then w solves the Cauchy problem

$$w''(x) = \gamma(x)w(x) \quad \text{for a.e. } x \in (0, 1),$$
$$w(x_0) = w'(x_0) = 0$$

and hence $w \equiv 0$, which contradicts (2.16). (2.17) is proved.
(ii) $y : [0, 1] \to [0, 1]$ is an increasing continuous map (for $dy/dx = (Lav^2e^B)^{-1} > 0$ a.e. in $(0, 1)$). Moreover, $y \in W^{1,1}(0, 1)$ (using (1.6) and (i)), and also $y^{-1} \in W^{1,1}(0, 1)$. (See (i) for

the proof of a similar result for z.)
(iii) To check that $\hat\rho \in L^p(0,1)$ when $1<p<\infty$, we use (1.8), (2.8)-(2.9) and (i) to get

$$\begin{aligned}\int_0^1 |\hat\rho(y)|^p dy &= \int_0^1 [L^2 a(x)v^4(x)e^{2B(x)}\rho(x)]^p \frac{dy}{dx}dx\\ &= L^{2p-1}\int_0^1 a^{p-1}\rho^p v^{4p-2}e^{(2p-1)B}dx\\ &< \infty.\end{aligned}$$

The fact that $\hat\rho\in L^\infty(0,1)$ when (1.8) holds with $p=\infty$ is obvious. On the other hand, $\hat\rho(y)>0$ for a.e. $y\in(0,1)$, for

$$\int_0^1 \chi_{\{\hat\rho(y)\le 0\}}(y)dy = \int_0^1 \chi_{\{(a\rho)(x)\le 0\}}(x)\frac{dy}{dx}dx = 0.$$

(iv) We first derive the PDE satisfied by u_2.

$$\begin{aligned}e^{-B}(av^2e^Bu_{2,x})_x &= e^{-B}\big(av^2e^B(\frac{u_{1,x}}{v}-\frac{u_1}{v^2}v_x)\big)_x\\ &= e^{-B}\big(ae^B(vu_{1,x}-v_xu_1)\big)_x\\ &= e^{-B}\big(v(ae^Bu_{1,x})_x - u_1(ae^Bv_x)_x\big)\\ &= v\rho u_{1,t}\\ &= \rho v^2 u_{2,t}\end{aligned}\tag{2.19}$$

(The first equality follows from (2.7), the third from basic algebra, the fourth from (1.1), (2.1)-(2.4) and (2.6), and the last from (2.7) again.) Since $\partial_y = (dx/dy)\partial_x = Lav^2e^B\partial_x$, (2.19) combined with (2.9) gives (2.10). (2.13) is obvious. It remains to establish (2.11)-(2.12). We focus on (2.11), (2.12) being obtained the same way. From the definition of u_2 we obtain

$$au_x = e^{Kt}a(v_xu_2+vu_{2,x}) \qquad \text{a.e. in } (0,1).\tag{2.20}$$

Combined with (1.2), this gives

$$\alpha_0u_2(0,t)+\beta_0((av_x)(0)u_2(0,t)+(au_{2,x})(0,t))=0.$$

On the other hand

$$\hat u_y = (dx/dy)u_{2,x} = La(x)v^2(x)e^{B(x)}u_{2,x}$$

and hence $\hat u_y(0,t) = L(au_{2,x})(0,t)$. Then (2.11) follows with

$$\hat\alpha_0 = \alpha_0+\beta_0(av_x)(0),\qquad \hat\beta_0 = L^{-1}\beta_0.$$

□

2.2. **Null controllability of the control problem (1.11)-(1.14).** Assume given $p\in(1,\infty]$, $\rho\in L^p(0,1)$ with $\rho(x)>0$ for a.e. $x\in(0,1)$, and $(\alpha_0,\beta_0),(\alpha_1,\beta_1)\in\mathbb{R}^2\setminus\{(0,0)\}$. Let $'=d/dx$, and let

$$L^2_\rho := \{f:(0,1)\to\mathbb{R}; ||f||^2_{L^2_\rho} := \int_0^1 |f(x)|^2\rho(x)dx<\infty\}.$$

Proposition 2.2. *Let $p,\rho,\alpha_0,\beta_0,\alpha_1,$ and β_1 be as above. Then there are a sequence $(e_n)_{n\ge0}$ in L^2_ρ and a sequence $(\lambda_n)_{n\ge0}$ in $\mathbb{R}$ such that*

(i) $(e_n)_{n\geq 0}$ *is an orthonormal basis in* L^2_ρ*;*

(ii) *For all* $n \geq 0$*,* $e_n \in W^{2,p}(0,1)$ *and* e_n *solves*

$$-e_n'' = \lambda_n \rho e_n \quad \text{in } (0,1), \tag{2.21}$$

$$\alpha_0 e_n(0) + \beta_0 e_n'(0) = 0, \tag{2.22}$$

$$\alpha_1 e_n(1) + \beta_1 e_n'(1) = 0. \tag{2.23}$$

(iii) *The sequence* $(\lambda_n)_{n\geq 0}$ *is strictly increasing, and for some constant* $C > 0$

$$\lambda_n \geq Cn \quad \text{for } n \gg 1. \tag{2.24}$$

Proof. Let us consider the elliptic boundary value problem

$$-u'' + \lambda^* \rho u = \rho f \quad \text{in } (0,1), \tag{2.25}$$

$$\alpha_0 u(0) + \beta_0 u'(0) = 0, \tag{2.26}$$

$$\alpha_1 u(1) + \beta_1 u'(1) = 0 \tag{2.27}$$

where $\lambda^* \gg 1$ will be chosen later on. Introduce the symmetric bilinear form

$$a(u,v) := \int_0^1 (u'v' + \lambda^* \rho uv)dx + a_b(u,v)$$

where

$$a_b(u,v) := \begin{cases} \frac{\alpha_1}{\beta_1} u(1)v(1) - \frac{\alpha_0}{\beta_0} u(0)v(0) & \text{if } \beta_1 \neq 0 \text{ and } \beta_0 \neq 0, \\ \frac{\alpha_1}{\beta_1} u(1)v(1) & \text{if } \beta_1 \neq 0 \text{ and } \beta_0 = 0, \\ -\frac{\alpha_0}{\beta_0} u(0)v(0) & \text{if } \beta_1 = 0 \text{ and } \beta_0 \neq 0, \\ 0 & \text{if } \beta_1 = 0 \text{ and } \beta_0 = 0. \end{cases}$$

Let

$$H := \{u \in H^1(0,1);\ u(0) = 0 \text{ if } \beta_0 = 0,\ u(1) = 0 \text{ if } \beta_1 = 0\}$$

be endowed with the $H^1(0,1)$-norm. Clearly, the form a is continuous on $H \times H$, for $H^1(0,1) \subset C^0([0,1])$ continuously. We claim that the form a is *coercive* if λ^* is large enough. We need the

Lemma 2.3. *For any* $\varepsilon > 0$*, there exists some number* $C_\varepsilon > 0$ *such that*

$$||u||^2_{L^\infty} \leq \varepsilon ||u'||^2_{L^2} + C_\varepsilon ||u||^2_{L^2_\rho} \qquad \forall u \in H^1(0,1). \tag{2.28}$$

Proof of Lemma 2.3. If (2.28) is false, then one can find a number $\varepsilon > 0$ and a sequence $(u_n)_{n\geq 1}$ in $H^1(0,1)$ such that

$$1 = ||u_n||^2_{L^\infty} > \varepsilon ||u_n'||^2_{L^2} + n||u_n||^2_{L^2_\rho} \quad \forall \geq 1. \tag{2.29}$$

Thus $||u_n||^2_{H^1} \leq 1 + \varepsilon^{-1}$, and for some subsequence (u_{n_k}) and some $u \in H^1(0,1)$ we have

$$u_{n_k} \to u \text{ weakly in } H^1(0,1). \tag{2.30}$$

Since $H^1(0,1) \subset C^0([0,1]) \subset L^2_\rho$ continuously, the first embedding being also compact, we infer that $u_{n_k} \to u$ in both $C^0([0,1])$ and L^2_ρ. Thus $||u||_{L^\infty} = 1$ by (2.29). But (2.29) yields also $u_n \to 0$ in L^2_ρ and hence $u = 0$, contradicting $||u||_{L^\infty} = 1$. Lemma 2.3 is proved. □

From (2.28), we infer the existence of some constants $C_1, C_2 > 0$ such that

$$C_1 ||u||^2_{H^1} \leq ||u'||^2_{L^2} + ||u||^2_{L^2_\rho} \leq C_2 ||u||^2_{H^1} \quad \forall u \in H^1(0,1). \tag{2.31}$$

Next, we have for some $C^* > 0$

$$|a_b(u,u)| \le C^*||u||^2_{L^\infty} \le C^*(\varepsilon||u'||^2_{L^2} + C_\varepsilon||u||^2_{L^2_\rho}) \le \frac{1}{2}(||u'||^2_{L^2} + \lambda^*||u||^2_{L^2_\rho}) \tag{2.32}$$

if we pick $0 < \varepsilon < (2C^*)^{-1}$ and $\lambda^* > 2C^*C_\varepsilon$. Then for all $u \in H^1(0,1)$ we have

$$a(u,u) \ge \frac{1}{2}(||u'||^2_{L^2} + \lambda^*||u||^2_{L^2_\rho}) \ge C||u||^2_{H^1},$$

with $C := \min(1,\lambda^*)C_1/2$, as desired.

Let $f \in L^2_\rho$ be given. The linear form $L(v) = \int_0^1 \rho f v dx$ being continuous on H, it follows from Lax-Milgram theorem that there exists a unique function $u \in H$ such that

$$a(u,v) = L(v) \quad \forall v \in H. \tag{2.33}$$

Taking any $v \in C_0^\infty(0,1)$ in (2.33), we infer that (2.25) holds in the distributional sense. Furthermore $u \in W^{2,1}(0,1)$. Next, multiplying each term in (2.25) by $v \in C^\infty([0,1]) \cap H$, integrating over $(0,1)$ and comparing with (2.33), we obtain (2.26)-(2.27).

The operator $T : f \in L^2_\rho \to u = T(f) \in L^2_\rho$ is continuous, compact, and self-adjoint. It is also positive definite, for

$$C||u||^2_{H^1} \le a(u,u) = (f,u)_{L^2_\rho} \quad \text{and} \quad u = 0 \iff f = 0.$$

By the spectral theorem, there are an orthonormal basis $(e_n)_{n\ge 0}$ in L^2_ρ and a sequence $(\mu_n)_{n\ge 0}$ in $(0,+\infty)$ with $\mu_n \searrow 0$ such that $T(e_n) = \mu_n e_n$ for all $n \ge 0$. Thus (2.21)-(2.23) hold with $\lambda_n = \mu_n^{-1} - \lambda^*$. The eigenfunction $e_n \in W^{2,p}(0,1)$ by (2.21) and the fact that $\rho \in L^p(0,1)$ and $e_n \in L^\infty(0,1)$.

(iii) The sequence $(\lambda_n)_{n\ge 0}$ is known to be nondecreasing. It is (strictly) increasing if each eigenvalue λ_n is simple, a fact which is easily established: if e and $\tilde{e}$ are two eigenfunctions associated with the same eigenvalue λ_n, then the Wronskian $W(x) := e(x)\tilde{e}'(x) - e'(x)\tilde{e}(x)$ satisfies $W'(x) = 0$ a.e. and $W(0) = 0$, and hence $W \equiv 0$ in $(0,1)$. It follows that e and $\tilde{e}$ are proportional.

Let us prove (2.24). Consider for any $\lambda \ge 1$ the system

$$\begin{aligned} -e'' &= \lambda\rho e, && (2.34)\\ \alpha_0 e(0) + \beta_0 e'(0) &= 0, && (2.35)\\ \alpha_1 e(1) + \beta_1 e'(1) &= 0. && (2.36)\end{aligned}$$

Following [5], we introduce the Prüfer substitution

$$\begin{aligned} e' &= r\cos\theta, && (2.37)\\ e &= r\sin\theta, && (2.38)\end{aligned}$$

so that

$$\begin{aligned} r^2 &= e'^2 + e^2, && (2.39)\\ \tan\theta &= \frac{e}{e'}\cdot && (2.40)\end{aligned}$$

Then (r, θ) satisfies

$$\frac{dr}{dx} = r(1-\lambda\rho)\cos\theta\sin\theta, \tag{2.41}$$

$$\frac{d\theta}{dx} = \cos^2\theta + \lambda\rho\sin^2\theta. \tag{2.42}$$

Conversely, if (r, θ) satisfies (2.41)-(2.42), then one readily sees that (2.34) and (2.37) hold.

The condition (2.35) is expressed in terms of θ as

$$\theta_{|x=0} = \theta_0 := \begin{cases} -\arctan(\frac{\beta_0}{\alpha_0}) & \text{if } \alpha_0 \neq 0, \\ \frac{\pi}{2} & \text{if } \alpha_0 = 0. \end{cases} \tag{2.43}$$

Denote by $\theta(x, \lambda)$ the solution of (2.42) and (2.43). (Note that r is not present in (2.42).) Introduce

$$\theta_1 := \begin{cases} -\arctan(\frac{\beta_1}{\alpha_1}) & \text{if } \alpha_1 \neq 0, \\ \frac{\pi}{2} & \text{if } \alpha_1 = 0. \end{cases} \tag{2.44}$$

Then (e, λ) is a pair of eigenfunction/eigenvalue if and only if

$$\theta(1, \lambda) = \theta_1 \mod \pi. \tag{2.45}$$

Since the map $(x, \theta, \lambda) \to \cos^2\theta + \lambda\rho(x)\sin^2\theta$ is integrable in x and of class C^1 in (θ, λ), it follows that the map $(x, \lambda) \to \theta(x, \lambda)$ is well defined and continuous for $x \in [0, 1]$ and $\lambda \geq 1$. On the other hand, since the map $\lambda \to \cos^2\theta + \lambda\rho(x)\sin^2\theta$ is strictly increasing for a.e. x (provided that $\theta \notin \pi\mathbb{Z}$), it follows from a classical comparison theorem (see e.g. [5]) that the map $\lambda \to \theta(1, \lambda)$ is strictly increasing.

Let

$$\bar\theta(x) := \lim_{\lambda\to\infty} \theta(x, \lambda), \qquad x \in [0, 1].$$

We claim that

$$\bar\theta(1) = \infty. \tag{2.46}$$

If (2.46) fails, then we have for all $x \in [0, 1]$ and all $\lambda \geq 1$

$$\theta_0 \leq \theta(x, \lambda) \leq \theta(1, \lambda) \leq \bar\theta(1) < \infty,$$

where we used the fact that the r.h.s. of (2.42) is positive a.e. Integrating in (2.42) over (a, b), where $0 \leq a < b \leq 1$, gives then

$$\theta(b, \lambda) - \theta(a, \lambda) = \int_a^b \cos^2\theta(x, \lambda)dx + \lambda \int_a^b \rho(x)\sin^2\theta(x, \lambda)dx. \tag{2.47}$$

An application of the Dominated Convergence Theorem yields

$$\int_a^b \cos^2\theta(x, \lambda)dx \to \int_a^b \cos^2\bar\theta(x)dx, \tag{2.48}$$

$$\int_a^b \rho(x)\sin^2\theta(x, \lambda)dx \to \int_a^b \rho(x)\sin^2\bar\theta(x)dx \tag{2.49}$$

as $\lambda \to \infty$.

Letting $\lambda \to \infty$ in (2.47) and using (2.48)-(2.49), we infer

$$\int_a^b \rho(x)\sin^2\bar\theta(x)dx = 0.$$

The numbers a and b being arbitrary, this shows that $\bar\theta(x) \in \pi\mathbb{Z}$ for a.e. $x \in (0,1)$. The function $\bar\theta$ being nondecreasing and bounded, it is piecewise constant. Choosing $a < b$ such that $\bar\theta$ is constant on $[a,b]$ and letting $\lambda \to \infty$ in (2.47), we obtain $0 \geq b-a$, which is a contradiction.

Thus (2.46) is established, and we see that for any $n \gg 1$ we can find a unique $\tilde\lambda_n \geq 1$ such that

$$\theta(1,\tilde\lambda_n) = \theta_1 + n\pi.$$

Then λ_n and $\tilde\lambda_n$ must agree, up to a translation in the indices, i.e. $\lambda_n = \tilde\lambda_{n-\bar n}$ for some $\bar n \in \mathbb{Z}$. Thus we can write

$$\theta(1,\lambda_n) = \theta_1 + (n-\bar n)\pi.$$

Integrating in (2.42), we obtain

$$\theta_1 + (n-\bar n)\pi - \theta_0 = \int_0^1 (\cos^2\theta + \lambda_n\rho\sin^2\theta)dx \leq 1 + \lambda_n\int_0^1 \rho(x)dx.$$

Since $\theta_0, \theta_1 \in (-\pi/2, \pi/2]$ and $\int_0^1 \rho(x)dx > 0$, (2.24) follows. □

Remark 2.4. *If, in addition, $\alpha_0\beta_0 \leq 0$ and $\alpha_1\beta_1 \geq 0$, then using a modified Prüfer system as in [5, 21] we can actually prove that*

$$\lambda_n \geq Cn^2 \qquad \textit{for}\ \ n \gg 1.$$

We now turn our attention to the generating functions g_i $(i \geq 0)$ defined along (1.20)-(1.25).

Proposition 2.5.

(i) $g_0(x) = (\alpha_0^2+\beta_0^2)^{-1}(\beta_0 - \alpha_0 x)$

(ii) *There are some constants $C, R > 0$ such that*

$$||g_i||_{W^{2,p}(0,1)} \leq \frac{C}{R^i(i!)^{2-\frac{1}{p}}} \qquad \forall i \geq 0\cdot \tag{2.50}$$

Proof. (i) is obvious. For (ii), we first notice that g_i may be written as

$$g_i(x) = \int_0^x \Big(\int_0^s \rho(\sigma)g_{i-1}(\sigma)d\sigma\Big)ds. \tag{2.51}$$

Let $q \in [1,\infty)$ be the conjugate exponent of p, i.e. $p^{-1}+q^{-1} = 1$. We need the following

Lemma 2.6. *Let $f \in L^\infty(0,1)$ and $g(x) = \int_0^x \big(\int_0^s \rho(\sigma)f(\sigma)d\sigma\big)ds$. If*

$$|f(x)| \leq Cx^r \textit{ for a.e. } \ x \in (0,1) \tag{2.52}$$

for some constants $C, r \geq 0$, then

$$|g(x)| \leq C\frac{||\rho||_{L^p}}{q^{\frac{1}{q}}}\frac{x^{r+\frac{1}{q}+1}}{(r+\frac{1}{q})^{\frac{1}{q}}(r+\frac{1}{q}+1)} \qquad \forall x \in [0,1]. \tag{2.53}$$

Proof of Lemma 2.6. From the Hölder inequality and (2.52), we have for all $s \in (0,1)$

$$\begin{aligned} |\int_0^s \rho(\sigma) f(\sigma) d\sigma| &\leq ||\rho||_{L^p(0,s)} ||f||_{L^q(0,s)} \\ &\leq C\|\rho||_{L^p(0,1)} \left(\frac{s^{rq+1}}{rq+1} \right)^{\frac{1}{q}} \end{aligned}$$

so that

$$|g(x)| \leq C||\rho||_{L^p(0,1)} \frac{x^{r+\frac{1}{q}+1}}{(rq+1)^{\frac{1}{q}}(r+\frac{1}{q}+1)} \qquad \forall x \in [0,1].$$

□

Iterated applications of Lemma 2.6 yield

$$\begin{aligned} |g_i(x)| &\leq ||g_0||_{L^\infty} \left(\frac{||\rho||_{L^p}}{q^{\frac{1}{q}}} \right)^i \frac{x^{i(\frac{1}{q}+1)}}{\prod_{j=1}^i \left(\frac{1}{q} + (j-1)(1+\frac{1}{q}) \right)^{\frac{1}{q}} \prod_{j=1}^i j(1+\frac{1}{q})} \\ &\leq ||g_0||_{L^\infty} \left(\frac{||\rho||_{L^p}}{q^{\frac{1}{q}}} \right)^i \frac{1}{\left(\frac{1}{q}(1+\frac{1}{q})^{i-1}(i-1)! \right)^{\frac{1}{q}} i!(1+\frac{1}{q})^i} \\ &\leq \frac{C}{R^i i!^{1+\frac{1}{q}}} \end{aligned}$$

if we pick $R < ||\rho||_{L^p}^{-1} q^{\frac{1}{q}} (1+\frac{1}{q})^{1+\frac{1}{q}}$ and $C \gg 1$. Since $1/q = 1 - 1/p$, we infer that

$$||g_i||_{L^\infty} \leq \frac{C}{R^i i!^{2-\frac{1}{p}}}$$

which, combined with (1.23), yields (2.50). □

Remark 2.7.

(1) *The power of* $i!$ *in the computations above is essentially sharp, since*

$$s^i i! \leq \prod_{j=1}^i (r+js) \leq s^i (i+1)!$$

for $0 \leq r \leq s$.

(2) *When* $p = 1$*, the estimate* $||g_i||_{L^\infty(0,1)} \leq C/(R^i\, i!)$ *is not sufficient to ensure the convergence of the series in* (1.16) *when* $f \in G^s([0,T])$ *with* $1 < s < 2$.

The fact that we can expand the eigenfunctions in terms of the generating functions is detailed in the following

Proposition 2.8. *There is some sequence* $(\zeta_n)_{n\geq 0}$ *of real numbers such that for all* $n \geq 0$

$$e_n = \zeta_n \sum_{i\geq 0} (-\lambda_n)^i g_i \qquad \text{in } W^{2,p}(0,1). \tag{2.54}$$

Furthermore, for some constant $C>0$, we have

$$|\zeta_n| \le C(1+|\lambda_n|^{\frac{3}{2}}) \quad \forall n\ge 0. \tag{2.55}$$

Proof. From (2.50), we infer that the series in (2.54) is absolutely convergent, hence convergent, in $W^{2,p}(0,1)$. Let $\tilde e := \zeta_n \sum_{i\ge 0}(-\lambda_n)^i g_i$, where $\zeta_n\in\mathbb{R}$. Then

$$\tilde e'' = \zeta_n \sum_{i\ge 1} (-\lambda_n)^i \rho g_{i-1} = -\lambda_n \rho \tilde e \text{ in } L^p(0,1),$$

where we used (1.20) and (1.23). (1.21) and (1.24)-(1.25) yield

$$\alpha_0 \tilde e(0) + \beta_0 \tilde e'(0) = 0.$$

On the other hand, using (1.22) and (1.24)-(1.25), we obtain

$$\beta_0\tilde e(0) - \alpha_0 \tilde e'(0) = \zeta_n\big(\beta_0 g_0(0) - \alpha_0 g_0'(0)\big) = \zeta_n.$$

Hence, if we pick

$$\zeta_n := \beta_0 e_n(0) - \alpha_0 e_n'(0), \tag{2.56}$$

we have that $E := e_n - \tilde e$ satisfies

$$\alpha_0 E(0) + \beta_0 E'(0) = \beta_0 E(0) - \alpha_0 E'(0) = 0$$

and hence $E(0) = E'(0) = 0$ which, when combined with $-E'' = \lambda_n \rho E$, yields $E\equiv 0$, i.e. $e_n = \tilde e$. Thus (2.54) holds with ζ_n as in (2.56). To estimate ζ_n, we remind that e_n satisfies $T(e_n) = \mu_n e_n$, and hence

$$\mu_n a(e_n,e_n) = \int_0^1 \rho\, e_n^2 dx = 1.$$

Since $a(e_n,e_n)\ge C||e_n||^2_{H^1}$, we infer that $||e_n||^2_{H^1}\le C\mu_n^{-1}$, and hence

$$|e_n(0)| + |e_n(1)| \le C||e_n||_{H^1} \le C(1+|\lambda_n|^{\frac{1}{2}}).$$

On the other hand, (2.21) yields

$$||e_n''||_{L^p} \le C|\lambda_n|\, ||\rho||_{L^p} ||e_n||_{H^1} \le C(1+|\lambda_n|^{\frac{3}{2}}).$$

Thus

$$|\zeta_n| \le C||e_n||_{W^{2,p}} \le C(1+|\lambda_n|^{\frac{3}{2}}).$$

□

Since $p>1$, for any $s\in(1,2-\frac{1}{p})$ and any $0<\tau<T$, one may pick a function $\varphi\in G^s([0,2T])$ such that

$$\varphi(t) = \begin{cases} 1 & \text{if } t\le\tau, \\ 0 & \text{if } t\ge T. \end{cases}$$

We are in a position to prove the null controllability of (1.11)-(1.14). Let $u_0\in L^2_\rho$. Since $(e_n)_{n\ge 0}$ is an orthonormal basis in L^2_ρ, we can write

$$u_0 = \sum_{n\ge 0} c_n\, e_n \quad \text{in } L^2_\rho \tag{2.57}$$

with $\sum_{n\ge 0}|c_n|^2<\infty$. Let

$$y(t):=\varphi(t)\sum_{n\ge 0}c_n\zeta_n e^{-\lambda_n t}\qquad \text{for } t\in[\tau,T] \tag{2.58}$$

and

$$u(x,t)=\left\{\begin{array}{ll}\sum_{n\ge 0}c_n e^{-\lambda_n t}e_n(x) & \text{if } 0\le t\le\tau,\\ \sum_{i\ge 0}y^{(i)}(t)g_i(x) & \text{if } \tau<t\le T.\end{array}\right. \tag{2.59}$$

The main result in this section is the following

Theorem 2.9. *Let $p\in(1,\infty]$, $\rho\in L^p(0,1)$ with $\rho(x)>0$ for a.e. $x\in(0,1)$, $T>0$, $\tau\in(0,T)$, and $(\alpha_0,\beta_0),(\alpha_1,\beta_1)\in\mathbb{R}^2\setminus\{(0,0)\}$. Let $u_0\in L^2_\rho$ be decomposed as in (2.57), let $s\in(1,2-1/p)$, and let y be as in (2.58). Then $y\in G^s([\tau,T])$, and the control*

$$h(t)=\left\{\begin{array}{ll}0 & \text{if } 0\le t\le\tau,\\ \sum_{i\ge 0}y^{(i)}(t)(\alpha_1 g_i(1)+\beta_1 g_i'(1)) & \text{if } \tau<t\le T.\end{array}\right. \tag{2.60}$$

is such that the solution u of (1.11)-(1.14) satisfies $u(.,T)=0$. Moreover u is given by (2.59), $h\in G^s([0,T])$, and $u\in C([0,T],L^2_\rho)\cap G^s([\varepsilon,T],W^{2,p}(0,1))$ for all $0<\varepsilon\le T$.

Proof. Let $\mathbb{C}_+:=\{z=t+ir;\ t>0,\ r\in\mathbb{R}\}$. We notice that the map $z\to\sum_{n\ge 0}c_n\zeta_n e^{-\lambda_n z}$ is analytic in $\mathbb{C}_+$. Indeed, by (2.24) and (2.55), the series is clearly uniformly convergent on any compact set in $\mathbb{C}_+$. It follows that the map $t\to\sum_{n\ge 0}c_n\zeta_n e^{-\lambda_n t}$ is (real) analytic in $(0,\infty)$, hence in $G^1([\tau,T])\subset G^s([\tau,T])$. Thus $y\in G^s([\tau,T])$ by a classical result (see e.g. [36, Theorem 19.7]).

Let $\bar u$ denote the function defined in the r.h.s. of (2.59). We first prove that $\bar u\in G^1([\varepsilon,\tau], W^{2,p}(0,1))$ for all $\varepsilon\in(0,\tau)$. We have for $k\in\mathbb{N}$ and $\varepsilon\le t\le\tau$,

$$\begin{array}{rcl}||\partial_t^k(c_n e^{-\lambda_n t}e_n)||_{W^{2,p}} &=& |c_n|\,|\lambda_n|^k e^{-\lambda_n t}||e_n||_{W^{2,p}}\\ &\le& C|c_n|(1+|\lambda_n|^{k+\frac32})e^{-|\lambda_n|\varepsilon}\\ &\le& C\dfrac{|c_n|}{n+1}(1+|\lambda_n|^{k+3})e^{-|\lambda_n|\varepsilon}\\ &\le& C\dfrac{|c_n|}{n+1}(1+\varepsilon^{-k-3}(k+3)!),\end{array}$$

where we used (2.24) and $x^k/k!\le e^x$ for $x>0$ and $k\in\mathbb{N}$. Thus, applying Cauchy-Schwarz inequality, we obtain for $k\in\mathbb{N}$, $\varepsilon\le t\le\tau$ and some $C,\delta>0$

$$||\partial_t^k u||_{W^{2,p}}\le\sum_{n\ge 0}||\partial_t^k(c_n e^{-\lambda_n t}e_n)||_{W^{2,p}}\le\frac{C}{\delta^k}k!$$

which gives that $\bar{u} \in G^1([\varepsilon,\tau], W^{2,p}(0,1))$. It is clear that $\bar{u} \in C([0,\tau], L^2_\rho)$. Let us check that $\bar{u}(x,\tau^-) = \bar{u}(x,\tau^+)$. We have that for all $x \in [0,1]$

$$\begin{aligned}
\bar{u}(x,\tau^-) &= \sum_{n\geq 0} c_n e^{-\lambda_n \tau} e_n(x) \\
&= \sum_{n\geq 0} c_n e^{-\lambda_n \tau} \zeta_n \sum_{i\geq 0} (-\lambda_n)^i g_i(x) && (2.61)\\
&= \sum_{i\geq 0} \big(\sum_{n\geq 0} c_n \zeta_n e^{-\lambda_n\tau} (-\lambda_n)^i \big) g_i(x) && (2.62)\\
&= \sum_{i\geq 0} y^{(i)}(\tau) g_i(x) && (2.63)\\
&= \bar{u}(x,\tau^+).
\end{aligned}$$

For (2.61) we used (2.54). For (2.62), we used Fubini's theorem for series, which is licit for

$$\begin{aligned}
\sum_{i,n\geq 0} |c_n \zeta_n e^{-\lambda_n\tau} \lambda_n^i g_i(x)| &\leq C \sum_{i,n\geq 0} \frac{|c_n|}{n+1} (1+|\lambda_n|^{i+3}) \frac{e^{-|\lambda_n|\tau}}{R^i i!^{2-\frac{1}{p}}} \\
&\leq C \sum_{i,n\geq 0} \frac{|c_n|}{n+1} (1+\tau^{-i-3}(i+3)!) \frac{1}{R^i i!^{2-\frac{1}{p}}} \\
&\leq C (\sum_{n\geq 0} \frac{|c_n|}{n+1}) (\sum_{i\geq 0} \frac{1+\tau^{-i-3}(i+3)!}{R^i i!^{2-\frac{1}{p}}}) \\
&< \infty.
\end{aligned}$$

Finally for (2.63), we just used the fact that $\varphi(\tau) = 1$ and $\varphi^{(i)}(\tau) = 0$ for $i \geq 1$. It remains to prove that $\bar{u} \in G^s([\tau,T], W^{2,p}(0,1))$. Since $y \in G^s([\tau,T])$, there are some constants $C,\rho > 0$ such that $|y^{(i)}(t)| \leq C(i!)^s/\rho^i$. It follows that for $t \in [\tau,T]$

$$\begin{aligned}
\sum_{i\geq 0} ||\partial_t^j \, [y^{(i)}(t) g_i] \, ||_{W^{2,p}} &= \sum_{i\geq 0} ||y^{(i+j)}(t) g_i||_{W^{2,p}} \\
&\leq C \sum_{i\geq 0} \frac{(i+j)!^s}{\rho^{i+j}} \frac{1}{R^i i!^{2-\frac{1}{p}}} \\
&\leq C (\frac{2^s}{\rho})^j \left(\sum_{i\geq 0} (\frac{2^s}{\rho R})^i \frac{1}{i!^{2-\frac{1}{p}-s}} \right) j!^s && (2.64)
\end{aligned}$$

where we used $(i+j)! \leq 2^{i+j} i! j!$. Note that the series converges in (2.64), since $s < 2 - \frac{1}{p}$. Thus $\bar{u} \in G^s([\tau,T], W^{2,p}(0,1))$. It is clear that (1.11) is satisfied by $\bar{u}$ in the distributional sense in $(0,1)\times(0,\tau)$ and in $(0,1)\times(\tau,T)$. In particular

$$\partial_t^j \bar{u}(x,\tau^+) = (\rho^{-1}\partial_x^2)^j \bar{u}(x,\tau^+) = (\rho^{-1}\partial_x^2)^j \bar{u}(x,\tau^-) = \partial_t^j \bar{u}(x,\tau^-),$$

for the two series in (2.59) coincide at $t = \tau$, hence so do their space derivatives. This shows that $\bar{u} \in G^s([\varepsilon,T], W^{2,p}(0,1))$ for all $\varepsilon \in (0,\tau)$, and that (1.11) holds for $\bar{u}$ in $(0,1)\times(0,T)$.

The function h defined in (2.60) satisfies (1.13) (with u replaced by $\bar{u}$), and hence $h \in G^s([0,T])$ (for $\bar{u} \in G^s([\varepsilon,T], W^{2,p}(0,1))$ and $W^{2,p}(0,1) \subset C^1([0,1]))$. (1.12) and (1.14) are clearly satisfied by $\bar{u}$, and hence the solution u of (1.11)-(1.14) is $\bar{u}$. Finally $u(.,T)=0$, for $y^{(i)}(T)=0$ for all $i \geq 0$. $\square$

2.3. **End of the proof of Theorem 1.1.** Let $a,b,c,\rho,K,p,\alpha_0,\beta_0,\alpha_1,\beta_1,T$, and τ be as in the statement of Theorem 1.1. Pick any $u_0 \in L^1_\rho$ and any $s \in (1, 2-1/p)$. Let u denote the solution of (1.1)-(1.4) for a given $h \in G^s([0,T])$. Define $v, y, \hat{\rho}$, and $\hat{u}(y,t)$ as in Section 2.1. Then $\hat{u}$ solves (2.10)-(2.13) with initial state $\hat{u}_0(y(x)) = u_0(x)/v(x)$. It may occur that $\hat{u}_0 \notin L^2_{\hat{\rho}}$. However, $\hat{u}_0 \in L^1_{\hat{\rho}}$, for

$$\int_0^1 |\hat{u}_0(y)|\hat{\rho}(y)dy = \int_0^1 |\hat{u}_0(y(x))|\hat{\rho}(y(x))|\frac{dy}{dx}|dx = L\int_0^1 |u_0(x)|v(x)e^{B(x)}\rho(x)dx < \infty.$$

From the proof of Lemma 2.3, we know that the bilinear form $a(u,v)$ is a scalar product in H whose induced Hilbertian norm is equivalent to the usual H^1-norm, so that H can be viewed as a Hilbert space for this scalar product. Then it is easy to see that

(i) $(\sqrt{\mu_n}e_n)_{n\geq 0}$ is an orthonormal basis in H;

(ii) If, for $a \in \mathbb{R}$, $\mathcal{H}^a$ denotes the completion of Span$(e_n;\ n \geq 0)$ for the norm

$$||\sum_{n\geq 0} c_n e_n||_a := \left(\sum_{n\geq 0} \mu_n^{-a}|c_n|^2\right)^{\frac{1}{2}},$$

then $\mathcal{H}^0 = L^2_{\hat{\rho}}$ and $\mathcal{H}^1 = H$;

(iii) Identifying $L^2_{\hat{\rho}}$ with its dual, we obtain the diagram

$$\mathcal{H}^1 = H \subset L^2_{\hat{\rho}} = (L^2_{\hat{\rho}})' \subset H' = \mathcal{H}^{-1}.$$

See e.g. [24, pp. 7-17] for details. Since for any $w \in H \subset L^\infty(0,1)$,

$$\int_0^1 |\hat{u}_0(y)w(y)|\hat{\rho}(y)dy \leq ||w||_{L^\infty}\int_0^1 |\hat{u}_0(y)|\hat{\rho}(y)dy \leq C||w||_H \int_0^1 |u_0(x)|\rho(x)dx,$$

we infer that $\hat{u}_0 \in H'$. Setting $c_n := \int_0^1 \hat{u}_0(y)e_n(y)\hat{\rho}(y)dy$ for $n \geq 0$, the series $\sum_{n=0}^\infty c_n e^{-\lambda_n t}e_n$ belongs to $C([0,T],H') \cap C((0,\tau], L^2_{\hat{\rho}})$ and it takes the value $\hat{u}_0$ at $t=0$. The solution $\hat{u}$ defined in (2.59) with y as in (2.58) solves (2.10)-(2.13) with the control input $\hat{h}(t)$ defined in (2.60). Then the pair

$$u(x,t) := e^{Kt}v(x)\hat{u}(y(x),t), \tag{2.65}$$
$$h(t) := e^{Kt}\hat{h}(t) \tag{2.66}$$

satisfies (1.1)-(1.4) and $u(.,T)=0$. Pick any $\varepsilon \in (0,T)$. Since $v, y \in W^{1,1}(0,1)$, $\hat{u} \in G^s([\varepsilon,T], W^{2,p}(0,1))$, and $\hat{h} \in G^s([0,T])$, we have that

$$u \in G^s([\varepsilon,T], W^{1,1}(0,1)), \quad h \in G^s([0,T]).$$

Finally, since by (2.8) and (2.20) we have

$$\tilde{a}u_x = e^{Kt}\big((\tilde{a}v_x)\hat{u}(y(x),t) + (Lv)^{-1}\hat{u}_y(y(x),t)\big)$$

and since $(Lv)^{-1}, \tilde{a}v_x \in W^{1,1}(0,1)$ and $\hat{u} \in G^s([\varepsilon,T], W^{2,p}(0,1))$, it follows that

$$\tilde{a}u_x, au_x \in G^s([\varepsilon,T], C^0([0,1]))$$

and that (1.2)-(1.3) are satisfied. The proof of Theorem 1.1 is complete. □

Remark 2.10. *Since the map $x \to y(x)$ is absolutely continuous and* strictly increasing *on $[0,1]$, and the map $y \to \hat{u}_y(y,t)$ is absolutely continuous on $[0,1]$ for all $t \in (0,T]$, we infer that $x \to \hat{u}_y(y(x),t)$ is absolutely continuous on $[0,1]$ for all $t \in (0,T]$. (See [6,* Ex. 5.8.59 p. 391]*.) Thus $au_x(.,t) \in W^{1,1}(0,1)$ for all $t \in (0,T]$.*

3. Proof of Theorem 1.3

We shall show that the first step in the proof of Theorem 1.1 (see Section 2.1) can be slightly modified to reduce (1.36)-(1.39) to the canonical form (1.11)-(1.14). Next, the conclusion of Theorem 1.3 will follow from Theorem 2.9. We distinguish two cases: (i) $0 \le \mu < 1/4$ (subcritical case) and (ii) $\mu = 1/4$ (critical case).
(i) Assume that $0 \le \mu < 1/4$. We relax (2.4)-(2.5) to the problem

$$v_{xx} + \frac{\mu}{x^2}v = 0, \qquad x \in (0,1), \tag{3.1}$$
$$v(x) > 0, \qquad x \in (0,1), \tag{3.2}$$
$$v^{-2} \in L^1(0,1). \tag{3.3}$$

The general solution of (3.1) is found to be

$$v(x) = C_1x^{r_1} + C_2x^{r_2}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants, and r_1, r_2 denote the roots of the equation $r^2 - r + \mu = 0$, namely

$$r_1 = \frac{1-\sqrt{1-4\mu}}{2} \in (0,\frac{1}{2}), \qquad r_2 = \frac{1+\sqrt{1-4\mu}}{2} \in (\frac{1}{2},\infty).$$

Then $v(x) := x^{r_1}$ satisfies (3.1)-(3.3).

From (1.36), we have that $\tilde{a} = a \equiv 1$, $B \equiv 0$. We set $u_1 := u$,

$$u_2(x,t) := \frac{u(x,t)}{v(x)}, \quad L := \int_0^1 v^{-2}(s)ds < \infty, \quad y(x) := L^{-1}\int_0^x v^{-2}(s)ds,$$

and

$$\hat{u}(y,t) := u_2(x,t), \quad \hat{\rho}(y) := L^2v^4(x).$$

Again, $y:[0,1]\to[0,1]$ is an increasing bijection with $y,y^{-1}\in W^{1,1}(0,1)$, and $\hat u$ satisfies

$$\hat u_{yy}-\hat\rho(y)\hat u_t = 0, \quad y\in(0,1),\ t\in(0,T), \tag{3.4}$$

$$\hat u(0,t) = 0, \quad t\in(0,T), \tag{3.5}$$

$$(\alpha_1+\beta_1 r_1)\hat u(1,t)+\frac{\beta_1}{L}\hat u_y(1,t) = h(t), \quad t\in(0,T), \tag{3.6}$$

$$\hat u_0(y,0) = \hat u_0(y) := \frac{u_0(x)}{v(x)}, \quad y\in(0,1)\cdot \tag{3.7}$$

Note that $\hat u_0\in L^2_{\hat\rho}$, for

$$\int_0^1 |\hat u_0(y)|^2\hat\rho(y)dy = L\int_0^1 |u_0(x)|^2dx<\infty.$$

On the other hand $\hat\rho\in L^\infty(0,1)$. By Theorem 2.9, there is some $h\in G^s([0,T])$ such that the solution $\hat u$ of (3.4)-(3.7) satisfies $\hat u(.,T)=0$. Furthermore

$$\hat u\in G^s([\varepsilon,T],W^{2,\infty}(0,1)). \tag{3.8}$$

The corresponding trajectory u satisfies (1.36)-(1.39) and $u(.,T)\equiv 0$. Finally, from the expressions

$$\begin{aligned} u &= vu_2 = v\hat u(y(x),t)\\ u_x &= v_x\hat u(y(x),t)+v\hat u_y(y(x),t)\frac{dy}{dx}, \end{aligned}$$

(3.8), and the explicit form of v, we readily see that $u\in G^s([\varepsilon,T],W^{1,1}(0,1))$ and $x^r u_x\in G^s([\varepsilon,T],C^0([0,1]))$ for $r>1-r_1=(1+\sqrt{1-4\mu})/2$ and $\varepsilon\in(0,1)$.
(ii) Assume now that $\mu=1/4$. Assume first that $\beta_1=0$. We notice that the general solution of (3.1) takes the form

$$v(x)=C_1\sqrt{x}\ln x+C_2\sqrt{x}.$$

Picking $v(x):=-\sqrt{x}\ln x$, we see that (3.1)-(3.3) are satisfied. Performing the same change of variables as in (i) (but with the new expression of v) and applying again Theorem 2.9, we infer the existence of $h\in G^s([0,T])$ such that the solution $\hat u$ of (3.4)-(3.7) satisfies $\hat u(.,T)=0$. The corresponding trajectory u satisfies (1.36)-(1.39) and $u(.,T)=0$. Furthermore, $u\in G^s([\varepsilon,T],W^{1,1}(0,1))\cap C^\infty([\varepsilon,1]\times[\varepsilon,T])$ (by using classical regularity results). For the general Robin-Neumann condition at $x=1$ it is sufficient to set $h(t):=\alpha_1 u(1,t)+\beta_1 u_x(1,t)$ with the trajectory u constructed above with the Dirichlet control at $x=1$. The proof of Theorem 1.3 is complete. □

As a possible application, we consider the boundary control by the flatness approach of radial solutions of the heat equation in the ball $B(0,1)\subset\mathbb{R}^N$ $(2\le N\le 3)$. Using the radial coordinate $r=|x|$, we thus consider the system

$$u_{rr}+\frac{N-1}{r}u_r-u_t = 0, \quad r\in(0,1),\ t\in(0,T), \tag{3.9}$$

$$u_r(0,t) = 0, \quad t\in(0,T), \tag{3.10}$$

$$\alpha_1 u(1,t)+\beta_1 u_r(1,t) = h(t), \quad t\in(0,T) \tag{3.11}$$

$$u(r,0) = u_0(r), \quad r\in(0,1). \tag{3.12}$$

Note that Theorem 1.1 cannot be applied directly to (3.9)-(3.12), for (1.7) fails. (Note that, in sharp contrast, the control on a ring-shaped domain $\{r_0 < |x| < r_1\}$ with $r_1 > r_0 > 0$ is fully covered by Theorem 1.1, the coefficients in (3.9) being then smooth and bounded.)

We use the following change of variables from [13] which allows to remove the term with the first order derivative in r in (3.9):

$$u(r,t) = \tilde{u}(r,t)\exp(-\frac{1}{2}\int_0^r \frac{N-1}{s}ds) = \frac{\tilde{u}(r,t)}{r^{\frac{N-1}{2}}}. \tag{3.13}$$

Then (3.9) becomes

$$\tilde{u}_{rr} + \frac{(N-1)(3-N)}{4}\frac{\tilde{u}}{r^2} - \tilde{u}_t = 0. \tag{3.14}$$

This equation has to be supplemented with the boundary/initial conditions

$$\tilde{u}(0,t) = 0, \quad t \in (0,T), \tag{3.15}$$

$$(\alpha_1 - \frac{N-1}{2}\beta_1)\tilde{u}(1,t) + \beta_1\tilde{u}_r(1,t) = h(t), \quad t \in (0,T), \tag{3.16}$$

$$\tilde{u}(r,0) = r^{\frac{N-1}{2}}u_0(r), \quad r \in (0,R). \tag{3.17}$$

For $N = 3$, (3.14) reduces to the simple heat equation $\tilde{u}_{rr} - \tilde{u}_t = 0$ to which Theorem 1.1 can be applied. In particular $\tilde{u} \in G^s([\varepsilon,T], W^{2,\infty}(0,1))$. Actually, it is well known that $\tilde{u} \in C^\infty([0,1]\times[\varepsilon,T])$, so that we can write a Taylor expansion

$$\tilde{u}(r,t) = r\tilde{u}_r(0,t) + \frac{r^3}{6}\tilde{u}_{rrr}(0,t) + O(r^4),$$

where we used the fact that $\tilde{u}(0,t) = \tilde{u}_{rr}(0,t) = 0$. This yields

$$u_r(0,t) = \frac{r}{3}\tilde{u}_{rrr}(0,t) + O(r^2),$$

so that (3.10) is fulfilled.

For $N = 2$, (3.14)-(3.17) is of the form (1.36)-(1.39) with $\mu = 1/4$. Therefore Theorem 1.3 can be applied to (3.14)-(3.17). Our concern now is the derivation of (3.10) when going back to the original variables. Recall that

$$u(r,t) = \frac{\tilde{u}(r,t)}{\sqrt{r}}, \quad v(r) = -\sqrt{r}\ln r, \quad y(r) = L^{-1}\int_0^r v^{-2}(s)ds, \quad \hat{u}(y,t) = \frac{\tilde{u}(r,t)}{v(r)} = -\frac{u(r,t)}{\ln r},$$

so that, with $dy/dr = (Lr\ln^2 r)^{-1}$,

$$u_r = -\frac{1}{r}\hat{u} - \frac{1}{Lr\ln r}\hat{u}_y.$$

This yields at fixed $t \in (0,T)$

$$\int_0^1 (|u|^2 + |u_r|^2)rdr \le \int_0^1 r\ln^2 r|\hat{u}(y(r))|^2 dr + C\int_0^1 \frac{\hat{u}^2}{r}dr + \int_0^1 \frac{|\hat{u}_y(y(r))|^2}{L^2 r\ln^2 r}dr =: I_1 + I_2 + I_3.$$

Since $\hat{u}(.,t) \in W^{2,\infty}(0,1)$, both I_1 and I_3 are finite. On the other hand, using $\hat{u}(0,t) = 0$, we obtain $|\hat{u}(y(r),t)| \le Cy(r) = \frac{C}{|\ln r|}$, and hence $I_2 < \infty$. Thus $\int_0^1(|u|^2 + |u_r|^2)rdr < \infty$, while for

$p \in (2, \infty)$

$$\int_0^1 |u_{rr} + \frac{u_r}{r}|^p r dr = \int_0^1 |u_t|^p r dr = \int_0^1 |\hat{u}_t(y(r), t)|^p r |\ln r|^p dr < \infty.$$

Thus the function $x \to u(|x|, t)$ belongs to $W^{2,p}(B(0,1)) \subset C^1(\overline{B(0,1)})$, so that (3.10) is satisfied.

4. Appendix: proof of Corollary 1.2

We apply first a reduction to a canonical form similar to (1.11)-(1.14) by doing exactly the same changes of variables as those described in Section 2.1. With $u_1, u_2, y, \hat{u}$, and $\hat{\rho}$ defined as in (2.6)-(2.9), we infer from (1.32) that

$$e^{-B}(av^2 e^B u_{2,x})_x = \rho v^2 u_{2,t} + v e^{-Kt} \chi_\omega f.$$

Multiplying each term in the above equation by $L^2 a v^2 e^{2B}$, and using the fact that $\partial_y = L a v^2 e^B \partial_x$, we arrive to

$$\hat{u}_{yy} = \hat{\rho}(y) \hat{u}_t + \chi_{\hat{\omega}} \hat{f},$$

where $\hat{\omega} = (\hat{l}_1, \hat{l}_2) := (y(l_1), y(l_2))$ and

$$\hat{f}(y(x), t) := L^2 a(x) v^3(x) e^{2B(x)} e^{-Kt} \chi_\omega(x) f(x, t).$$

Let $\hat{u}_0(y(x)) := u_0(x)/v(x)$. Pick $\hat{l}_1', \hat{l}_2'$ such that $\hat{l}_1 < \hat{l}_1' < \hat{l}_2' < \hat{l}_2$, and a function $\varphi \in C^\infty([0,1])$ such that $\varphi(y) = 1$ for $0 \le y \le \hat{l}_1'$ and $\varphi(y) = 0$ for $\hat{l}_2' \le y \le 1$. Applying Theorem 1.1, we can find two functions $h^1, h^2 \in G^s([0,T])$ such that the solutions $\hat{u}^1, \hat{u}^2$ of the following systems

$$\begin{aligned} \hat{u}^1_{yy} - \hat{\rho}(y)\hat{u}^1_t &= 0, \quad y \in (0,1), \ t \in (0,T), && (4.1)\\ \hat{\alpha}_0 \hat{u}^1(0,t) + \hat{\beta}_0 \hat{u}^1_y(0,t) &= 0, \quad t \in (0,T), && (4.2)\\ \hat{u}^1_y(1,t) &= h^1(t), \quad t \in (0,T), && (4.3)\\ \hat{u}^1(y,0) &= \hat{u}_0(y), \quad y \in (0,1), && (4.4) \end{aligned}$$

and

$$\begin{aligned} \hat{u}^2_{yy} - \hat{\rho}(y)\hat{u}^2_t &= 0, \quad y \in (0,1), \ t \in (0,T), && (4.5)\\ \hat{u}^2_y(0,t) &= h^2(t), \quad t \in (0,T), && (4.6)\\ \hat{\alpha}_1 \hat{u}^2(1,t) + \hat{\beta}_1 \hat{u}^2_y(1,t) &= 0, \quad t \in (0,T), && (4.7)\\ \hat{u}^2(y,0) &= \hat{u}_0(y), \quad y \in (0,1), && (4.8) \end{aligned}$$

satisfy

$$\hat{u}^1(y,T) = \hat{u}^2(y,T) = 0 \quad \text{for all} \quad y \in [0,1].$$

Then it is sufficient to set

$$\begin{aligned} \hat{u}(y,t) &:= \varphi(y)\hat{u}^1(y,t) + (1-\varphi(y))\hat{u}^2(y,t), && (4.9)\\ \hat{f}(y,t) &:= \varphi''(y)(\hat{u}^1(y,t) - \hat{u}^2(y,t)) + 2\varphi'(y)(\hat{u}^1_y(y,t) - \hat{u}^2_y(y,t)). && (4.10) \end{aligned}$$

Note that $\hat{f}$ is supported in $[\hat{l}_1', \hat{l}_2'] \times [0,T]$, with $\hat{f} \in G^s([\varepsilon, T], W^{1,1}(0,1))$ for all $\varepsilon \in (0,T)$, and that $\hat{u}$ solves

$$\hat{u}_{yy} - \hat{\rho}(y)\hat{u}_t = \chi_{\hat{\omega}}\hat{f}, \quad y \in (0,1),\ t \in (0,T), \tag{4.11}$$

$$\hat{\alpha}_0\hat{u}(0,t) + \hat{\beta}_0\hat{u}_y(0,t) = 0, \quad t \in (0,T), \tag{4.12}$$

$$\hat{\alpha}_1\hat{u}(1,t) + \hat{\beta}_1\hat{u}_y(1,t) = 0, \quad t \in (0,T), \tag{4.13}$$

$$\hat{u}(y,0) = \hat{u}_0(y), \quad y \in (0,1), \tag{4.14}$$

$$\hat{u}(y,T) = 0, \quad y \in (0,1). \tag{4.15}$$

Let

$$f(x,t) := \left(L^2 a(x) v^3(x) e^{2B(x)} e^{-Kt}\right)^{-1} \hat{f}(y(x),t).$$

Then f is supported in $[y^{-1}(\hat{l}_1'), y^{-1}(\hat{l}_2')] \times [0,T] \subset \omega \times [0,T]$. We claim that $f \in L^2(0,T,L^2_a(\omega))$. Indeed, we have that

$$\begin{aligned}
\int_0^T\int_\omega |f(x,t)|^2 a(x)dxdt &\le C\int_0^T\int_0^1 \chi_\omega(x)(L^3a(x)v^4(x)e^{3B(x)}e^{-2Kt})|f(x,t)|^2dxdt \\
&= C\int_0^T\int_0^1 \chi_{\hat{\omega}}(y(x))|\hat{f}(y(x),t)|^2|\frac{dy}{dx}|dxdt \\
&= C\int_0^T\int_{\hat{\omega}} |\hat{f}(y,t)|^2dydt,
\end{aligned}$$

and the last integral is finite, since $\hat{f}$ is given by (4.10) and $\hat{u}^1, \hat{u}^2 \in L^2(0,T,H^1(0,1))$. For $\hat{u}^1$, this can be seen by scaling (4.1) by $\hat{u}^1$, integrating over $(0,1) \times (0,t)$ for $0 < t \le T$, and using Gronwall's lemma combined with Lemma 2.3. Thus $f \in L^2(0,T,L^2_a(\omega))$. Let

$$u(x,t) := e^{Kt}v(x)\hat{u}(y(x),t).$$

Then u solves (1.32)-(1.35) and $u(.,T) = 0$. □

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